

Dark matter as a geometric effect in $f(R)$ gravity

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We consider the behavior of the tangential velocity of test particles moving in stable circular orbits in $f(R)$ modified theories of gravity. A large number of observations at the galactic scale have shown that the rotational velocities of massive test particles (hydrogen clouds) tend towards constant values at large distances from the galactic center. We analyze the vacuum gravitational field equations in $f(R)$ models in the constant velocity region, and the general form of the metric tensor is derived in a closed form. The resulting modification of the Einstein-Hilbert Lagrangian is of the form R^{1+n} , with the parameter n expressed in terms of the tangential velocity. Therefore we find that to explain the motion of test particles around galaxies requires only very mild deviations from classical general relativity, and that modified gravity can explain the galactic dynamics without the need of introducing dark matter.

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I. INTRODUCTION

Modern astrophysical and cosmological models are faced with two severe theoretical difficulties, that can be summarized as the dark energy and the dark matter problems. Despite the fact that several suggestions have recently been proposed to overcome these issues, a satisfactory answer is yet to be obtained. However, in the context of dark matter, two observations, namely, the behavior of the galactic rotation curves and the mass discrepancy in galactic clusters, suggest the existence of a (non or weakly interacting) form of dark matter at galactic and extra-galactic scales.

The galactic rotation curves of spiral galaxies [1] are probably the most striking evidences for the possible failure of Newtonian gravity and of the general theory of relativity on galactic and intergalactic scales. In these galaxies, neutral hydrogen clouds are observed at large distances from the center, much beyond the extent of the luminous matter. As these clouds are moving in circular orbits with nearly constant tangential velocity v_{tg} , such orbits are maintained by the balance between the centrifugal acceleration v_{tg}^2/r and the gravitational attraction $GM(r)/r^2$ of the total mass $M(r)$ contained within

the radius r . This yields an expression for the galactic mass profile of the form $M(r) = rv_{\text{tg}}^2/G$, with the mass increasing linearly with r , even at large distances, where very little luminous matter has been detected [1]. This peculiar behavior of the rotation curves is usually explained by postulating the existence of dark matter, assumed to be a cold and pressureless medium, distributed in a spherical halo around the galaxies.

There are many possible candidates for dark matter, the most popular ones being the weakly interacting massive particles (WIMP). Their interaction cross sections with normal baryonic matter, although extremely small, are expected to be non-zero, and therefore it is believed to detect them directly [2]. However, no direct (non-gravitational) evidence for the existence of dark matter has been reported so far. It is important to emphasize that dark matter consisting of WIMP's may exist in the form of an Einstein cluster [3], or could possibly undergo a phase transition to form a Bose-Einstein condensate [4]. One cannot also *a priori* exclude the possibility that Einstein's (and Newton's) theory of gravity breaks down at galactic scales. In this context, several theoretical models, based on a modification of Newton's law or of general relativity, have been proposed so far to explain the behavior of the galactic rotation curves [5, 6, 7].

A promising avenue that has been extensively investigated recently are the $f(R)$ modified theories of gravity, where the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar R [8]. In this work we shall use the metric formalism, which consists

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in varying the action with respect to $g_{\mu\nu}$, although other alternative approaches have been considered in the literature, namely, the Palatini formalism [9, 10], where the metric and the connections are treated as separate variables; and the metric-affine formalism, where the matter part of the action now depends and is varied with respect to the connection [10]. It has been suggested that these modified gravity models account for the late time acceleration of the universe [11], thus challenging the need for dark energy. However, the viability of the $f(R)$ models proposed in the literature has been extensively analyzed [12, 13, 14]. In this context, severe weak field constraints in the solar system range seem to rule out most of the models proposed so far [15, 16, 17, 18], although viable models do exist [13, 19, 20, 21].

In order to be a viable theory, in addition to satisfying the solar system constraints, the proposed models should simultaneously account for the four distinct cosmological phases, namely, inflation, the radiation-dominated and matter-dominated epochs, and the late-time accelerated expansion [22], and be consistent with cosmological structure formation observations [23]. The issue of stability [24] also plays an important role in the viability of cosmological solutions [20, 21, 25]. It is interesting to note that, recently, viable cosmological $f(R)$ models were analyzed, and it was found that the latter models satisfying cosmological and local gravity constraints are practically indistinguishable from the Λ CDM model, at least at the background level [21].

The possibility that the galactic dynamics of massive test particles may be understood without the need for dark matter was also considered in the framework of $f(R)$ modified theories of gravity [26, 27, 28]. In the context of galactic dynamics, a version of $f(R)$ gravity models admitting a modified Schwarzschild-de Sitter metric was analyzed in [29]. In the weak field limit one obtains a small logarithmic correction to the Newtonian potential, and a test star moving in such a spacetime acquires a constant asymptotic speed at large distances. It is interesting to note that the model has similar properties with MOND [5]. A model based on a generalized action with $f(R) = R + R(R/R_0 + 2/\alpha)^{-1} \ln(R/R_c)$, where α , R_0 and R_c are constants, was proposed in [30]. In particular, this model can describe the Pioneer anomaly and the flat rotation curves of the spiral galaxies. In a cosmological context, the vacuum solution also results in a late time acceleration for the universe. The generalization of the virial theorem in $f(R)$ modified gravity, using the collisionless Boltzmann equation, was considered in [31].

Within the framework of $f(R)$ gravity, a model exhibiting an explicit coupling of an arbitrary function of R with the matter Lagrangian density was proposed recently [32]. Due to this coupling a connection between the problem of the rotation curve of galaxies, via a solution somewhat similar to the one put forward in the context of MOND, and the Pioneer anomaly is established.

It is the purpose of the present paper to consider, from an exact analytic point of view, the problem of the galactic rotation curves in the framework of $f(R)$ modified theories of gravity. In order to find an exact analytic description of the galactic dynamics of test particles in $f(R)$ gravity models, we start from the general relativistic expression of the tangential velocity v_{tg} of massive test particles in static and spherically symmetric spacetimes, moving in stable circular orbits around the galactic center. The rotational velocity is determined by the g_{tt} component of the metric tensor and of the radial distance only.

We limit our analysis to the most important region for the galactic dynamics of test particles, namely, the region of constant rotational velocities. The constancy of the tangential velocity completely determines the form of g_{tt} , and consequently, the exact analytical form of the latter metric tensor component is completely determined from dynamical considerations.

As a next step in our analysis of the geometry in the constant velocity region we consider the spherically symmetric vacuum solutions of the gravitational field equations in $f(R)$ modified theories of gravity. By introducing several coordinate and functional transformations, the field equations can be reduced to an autonomous system of differential equations. By using the general form of g_{tt} we obtain a second order differential equation fixing the functional form of g_{rr} in the constant velocity region. This equation has as an exact solution $g_{rr} = \text{constant}$, which gives g_{rr} as a function of the observed tangential velocity only. The expressions of the Ricci scalar and of the function $f(R)$ are also obtained.

As a general conclusion of our study we find that to explain the flat galactic rotation curves, only small deviations from standard general relativity are needed, so that $f(R) \propto R^{1+v_{\text{tg}}^2}$, a somewhat natural result in the context of modified gravity theories.

As a possible observational test of our results we suggest the study of the lensing of light by galaxies in the constant velocity region. The study of lensing may, in principle, discriminate between the present model and other dark matter models. On the other hand, the deflection angle in our model is of the same order of magnitude as the deflection angle in the standard isothermal sphere dark matter model, which is well tested observationally. This shows that the results obtained in this paper are consistent both theoretically and observationally.

The present paper is organized as follows. The tangential velocity of a test particle in modified theories of gravity is derived in Section II. The vacuum field equations in the $f(R)$ models are written down in Section III. Two specific solutions of the field equations in the constant tangential velocity region are presented in Section IV, where some general properties of the basic equation describing the behavior of g_{rr} are also discussed. We discuss and conclude our results in Section V.

II. THE MOTION OF MASSIVE TEST PARTICLES IN STABLE CIRCULAR ORBITS

In order to obtain results which are relevant to the galactic dynamics, in the following, we restrict our study to the static and spherically symmetric metric given by

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. In the present paper, we use a system of units so that $G = c = 1$. The metric tensor coefficient $e^{\lambda(r)}$ is constrained by the condition $e^{\lambda(r)} \geq 1, \forall r \in [0, \infty)$, while we assume that $e^{\nu(r)}$ satisfies the condition $e^{\nu(r)} \geq 0, \forall r \in [0, \infty)$.

The Lagrangian \mathcal{L} for a massive test particle reads

$$\mathcal{L} = \frac{1}{2} \left(-e^{\nu} \dot{t}^2 + e^{\lambda} \dot{r}^2 + r^2 \dot{\Omega}^2 \right), \quad (2)$$

where the overdot denotes differentiation with respect to the affine parameter s . Since the metric tensor coefficients do not explicitly depend on t and Ω , the Lagrangian (2) yields the following conserved quantities (generalized momenta) [33]:

$$-e^{\nu(r)} \dot{t} = E, \quad r^2 \dot{\Omega} = L, \quad (3)$$

where E is related to the total energy of the particle and L to the total angular momentum. With the use of the conserved quantities, we obtain from Eq. (2) the geodesic equation for massive particles in the form

$$e^{\nu+\lambda} \dot{r}^2 + e^{\nu} \left(1 + \frac{L^2}{r^2} \right) = E^2. \quad (4)$$

For the case of the motion of particles in circular and stable orbits the effective potential must satisfy the following conditions: a) $\dot{r} = 0$, representing circular motion; b) $\partial V_{eff}/\partial r = 0$, providing extreme motion; c) $\partial^2 V_{eff}/\partial r^2|_{extr} > 0$, translating a stable orbit [33]. Conditions a) and b) immediately provide the conserved quantities as

$$E^2 = e^{\nu} \left(1 + \frac{L^2}{r^2} \right), \quad (5)$$

and

$$\frac{L^2}{r^2} = \frac{r\nu'}{2} e^{-\nu} E^2, \quad (6)$$

respectively. Equivalently, these two equations can be rewritten as

$$E^2 = \frac{e^{\nu}}{1 - r\nu'/2}, \quad L^2 = \frac{r^3 \nu'/2}{1 - r\nu'/2}. \quad (7)$$

We define the tangential velocity v_{tg} of a test particle, as measured in terms of the proper time [34], that is, by an observer located at the given point, as

$$v_{tg}^2 = e^{-\nu} r^2 \left(\frac{d\Omega}{dt} \right)^2 = e^{-\nu} r^2 \dot{\Omega}^2 / \dot{t}^2 = e^{\nu} \frac{L^2}{r^2 E^2}. \quad (8)$$

By using the constants of motion, we obtain the expression of the tangential velocity of a test particle in a stable circular orbit [33], given by

$$v_{tg}^2 = \frac{r\nu'}{2}. \quad (9)$$

This simple expression which relates one of the metric components to the tangential velocity has three important properties. First of all, it is an exact general relativistic expression valid for static and spherically symmetric spacetimes. Secondly, it is interesting to note that the tangential velocity is sensitive to only one of the two metric functions, i.e., it is independent of the form of g_{rr} . Lastly, since the motion of test particles is defined via the geodesic equations, this relation is independent of $f(R)$. Even if we allow the modified theory of gravity to contain arbitrary contractions of the Ricci and Riemann tensors, the above equations would still hold exactly.

In regions with constant tangential velocity the metric function ν is fixed by the condition $v_{tg} \approx \text{constant}$, and by integrating Eq. (9), we find

$$\nu = 2v_{tg}^2 \ln \left(\frac{r}{r_0} \right), \quad (10)$$

where r_0 is a constant of integration. Therefore the most general static and spherically symmetric metric in the constant tangential velocity regions can be written as

$$ds^2 = - \left(\frac{r}{r_0} \right)^{2v_{tg}^2} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2. \quad (11)$$

Note, however, that the galactic rotation curves generally show a more complicated dynamics [1]. Therefore, to obtain a more accurate and realistic description of the particle's motion, which can be extended beyond the constant velocity region, more general expressions of the metric coefficients are needed. However, in the present paper we restrict our analysis to the constant velocity region, which provides interesting results.

In order to test the consistency of our results, we consider the Newtonian limit of the model. The assumption of small velocities of the particles requires that the gravitational field be weak. In the Newtonian limit the g_{tt} component of the metric tensor is given by $e^{\nu} \approx 1 + 2\Phi_N$, where Φ_N is the Newtonian gravitational potential satisfying the Poisson equation $\Delta\Phi_N = 4\pi\rho$ [34]. In the constant velocity region the mass $M(r)$ of the dark matter and the energy density ρ vary with the distance as $M(r) = v_{tg}^2 r$ and $\rho = v_{tg}^2 / 4\pi r^2$, respectively. Therefore the Poisson equation is given by

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi_N}{dr} \right) = \frac{v_{tg}^2}{r^2}, \quad (12)$$

and has the general solution

$$\Phi_N(r) = v_{tg}^2 \ln \frac{r}{r_0} - \frac{C_N}{r}, \quad (13)$$

where C_N and r_0 are arbitrary constants of integration. In the limit of large r , corresponding to the constant velocity regions around galaxies, the Newtonian potential is given by

$$\Phi_N(r) \approx v_{\text{tg}}^2 \ln \frac{r}{r_0}, \quad (14)$$

reflecting a logarithmic dependence on the radial distance r . On the other hand, the g_{tt} component of the metric tensor can be represented in the constant velocity “dark matter” region as

$$\begin{aligned} e^\nu &\approx \left(\frac{r}{r_0}\right)^{2v_{\text{tg}}^2} = \exp \left[\ln \left(\frac{r}{r_0}\right)^{2v_{\text{tg}}^2} \right] \\ &\approx 1 + 2v_{\text{tg}}^2 \ln \left(\frac{r}{r_0}\right) = 1 + 2\Phi_N(r). \end{aligned} \quad (15)$$

Therefore the model has a well-defined Newtonian limit, and the metric given by Eq. (11) can indeed be used to describe the geometry of the spacetime in the dark matter dominated regions.

III. VACUUM FIELD EQUATIONS IN $f(R)$ GRAVITY

The action for the modified theories of gravity considered in this work takes the following form

$$S = \int f(R) \sqrt{-g} d^4x, \quad (16)$$

where $f(R)$ is an arbitrary analytical function of the Ricci scalar R . Note that we are only interested in the vacuum case, and therefore we have not added a matter Lagrangian to the action.

Varying the action with respect to the metric $g_{\mu\nu}$ yields the following field equations

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) F(R) = 0, \quad (17)$$

where we have denoted $F(R) = df(R)/dR$. Note that the covariant derivative of these field equations vanishes for all $f(R)$ by means of the generalized Bianchi identities [32, 35].

For a static and spherically symmetric metric of the form given by Eq. (1), the field equations of the $f(R)$ gravity in vacuum can be expressed as [29, 36]

$$F'' - \frac{1}{2}(\nu' + \lambda') F' - \frac{(\nu' + \lambda')}{r} F = 0, \quad (18)$$

$$\begin{aligned} \nu'' + \nu'^2 - \frac{1}{2}(\nu' + \lambda') \left(\nu' + \frac{2}{r} \right) - \frac{2}{r^2} (1 - e^\lambda) \\ = -2 \frac{F''}{F} + \left(\lambda' + \frac{2}{r} \right) \frac{F'}{F}, \end{aligned} \quad (19)$$

$$f = F e^{-\lambda} \left[\nu'' - \frac{1}{2}(\nu' + \lambda') \nu' - \frac{2}{r} \lambda' + \left(\nu' + \frac{4}{r} \right) \frac{F'}{F} \right], \quad (20)$$

$$R = 2 \frac{f}{F} - 3e^{-\lambda} \left\{ \frac{F''}{F} + \left[\frac{1}{2}(\nu' - \lambda') + \frac{2}{r} \right] \frac{F'}{F} \right\}. \quad (21)$$

It is useful to introduce a new variable η by means of the following transformation

$$\eta = \ln r. \quad (22)$$

Therefore, the field equations Eqs. (18)–(21) take the form

$$\frac{d^2 F}{d\eta^2} - \left[1 + \frac{1}{2} \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \right] \frac{dF}{d\eta} - \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) F = 0, \quad (23)$$

$$\begin{aligned} \frac{d^2 \nu}{d\eta^2} - \frac{d\nu}{d\eta} + \left(\frac{d\nu}{d\eta} \right)^2 - \frac{1}{2} \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \left(\frac{d\nu}{d\eta} + 2 \right) \\ + 2(1 - e^\lambda) = -2 \frac{1}{F} \frac{d^2 F}{d\eta^2} + \left(\frac{d\lambda}{d\eta} + 4 \right) \frac{1}{F} \frac{dF}{d\eta}, \end{aligned} \quad (24)$$

$$\begin{aligned} f = F e^{-\lambda - 2\eta} \left[\frac{d^2 \nu}{d\eta^2} - \frac{d\nu}{d\eta} - \frac{1}{2} \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \frac{d\nu}{d\eta} \right. \\ \left. - 2 \frac{d\lambda}{d\eta} + \left(\frac{d\nu}{d\eta} + 4 \right) \frac{1}{F} \frac{dF}{d\eta} \right], \end{aligned} \quad (25)$$

$$\begin{aligned} R = 2 \frac{f}{F} - 3e^{-\lambda - 2\eta} \left\{ \frac{1}{F} \frac{d^2 F}{d\eta^2} - \frac{1}{F} \frac{dF}{d\eta} \right. \\ \left. + \left[\frac{1}{2} \left(\frac{d\nu}{d\eta} - \frac{d\lambda}{d\eta} \right) + 2 \right] \frac{1}{F} \frac{dF}{d\eta} \right\}. \end{aligned} \quad (26)$$

As a result of introducing the new variable the basic field equations Eqs. (23) and (24) are independent of the radial coordinate η . It is also very useful to introduce a formal representation of the function F as

$$F(\eta) = F_0 \exp \left[\int u(\eta) d\eta \right], \quad (27)$$

where u is a new function of η , and F_0 is an arbitrary constant, so that $(1/F)dF/d\eta = u$, $(1/F)d^2F/d\eta^2 = du/d\eta + u^2$. Hence Eq. (23) can be written as

$$\frac{du}{d\eta} + u^2 - \left[1 + \frac{1}{2} \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) \right] u - \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta} \right) = 0. \quad (28)$$

This equation is a Riccati type first order differential equation. To find its general solution the knowledge of a

particular solution is required. By using the function u , Eq. (24) can be written as

$$\frac{d^2\nu}{d\eta^2} - \frac{d\nu}{d\eta} + \left(\frac{d\nu}{d\eta}\right)^2 - \frac{1}{2}\left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta}\right)\left(\frac{d\nu}{d\eta} + 2\right) + 2(1 - e^\lambda) = -2\frac{du}{d\eta} - 2u^2 + \left(\frac{d\lambda}{d\eta} + 4\right)u. \quad (29)$$

Substituting the term $du/d\eta + u^2$ in Eq. (29), with the use of Eq. (28), we obtain

$$2\left(1 - \frac{1}{2}\frac{d\nu}{d\eta}\right)u = 2(1 - e^\lambda) + \frac{d^2\nu}{d\eta^2} - \frac{d\nu}{d\eta} + \left(\frac{d\nu}{d\eta}\right)^2 + \left(\frac{d\nu}{d\eta} + \frac{d\lambda}{d\eta}\right)\left(1 - \frac{1}{2}\frac{d\nu}{d\eta}\right). \quad (30)$$

The general solution of Eqs. (28) and (30) provides the general static and vacuum solution of the modified field equations for the case of the $f(R)$ gravity models. Once $\nu(\eta)$ and $\lambda(\eta)$ are specified, one can immediately obtain u and consequently (by integration) F , as well as all the other relevant physical quantities. If the function F and the metric tensor coefficients are known, f can be obtained as a function of R from Eqs. (25) and (26) in a parametric form, as $f = f(\eta)$, $R = R(\eta)$.

IV. DARK MATTER AS A GEOMETRIC EFFECT IN $f(R)$ MODELS

In order to obtain a geometric interpretation of dark matter in $f(R)$ gravity, we start with the metric coefficients ν and λ in the dark matter dominated region at the galactic scale, given by Eq. (11), which, using the variable η , can be represented in the following form

$$\nu = 2m(\ln \eta - \ln \eta_0), \quad \lambda = \lambda(\eta), \quad (31)$$

where $\eta_0 = \ln r_0 = \text{constant}$, and for notational simplicity we have denoted

$$m = v_{\text{tg}}^2 = \text{constant}. \quad (32)$$

For this form of the metric Eqs. (28) and (30) become

$$\frac{du}{d\eta} + u^2 - \left(1 + m + \frac{1}{2}\frac{d\lambda}{d\eta}\right)u - 2m - \frac{d\lambda}{d\eta} = 0, \quad (33)$$

and

$$2(1 - m)u = 2(1 - e^\lambda) + 2m^2 + (1 - m)\frac{d\lambda}{d\eta}, \quad (34)$$

respectively. By taking the derivative with respect to η of Eq. (34) and substituting the resulting $du/d\eta$ and u in Eq. (33) provides the following second order differential

equation

$$\frac{1}{2}\frac{d^2\lambda}{d\eta^2} - \frac{1}{1 - m}\left(\frac{3}{2}e^\lambda - m^2 + 1 - m\right)\frac{d\lambda}{d\eta} + \frac{1}{(1 - m)^2}(1 - e^\lambda)^2 + \frac{3m^2 - 1}{(1 - m)^2}(1 - e^\lambda) + \frac{m^2(2m^2 - 1)}{(1 - m)^2} - 2m = 0, \quad (35)$$

which must be satisfied by the metric coefficient $\lambda(\eta)$ in the “dark matter” dominated regions in $f(R)$ gravity models.

From astrophysical observations it is known that the tangential velocity of test particles in circular stable orbits around the galactic center is of the order of $v_{\text{tg}} \approx 200 - 300$ km/s [1]. Hence, we have $m = v_{\text{tg}}^2 \approx 10^{-6}$, and all the terms containing m^2 and m^4 in Eq. (35) can be neglected within a very good approximation. Considering, thus, a first order approximation in m , Eq. (35) describing the metric coefficient $\exp(\lambda)$ reduces to

$$\frac{1}{2}\frac{d^2\lambda}{d\eta^2} - \frac{1}{1 - m}\left(\frac{3}{2}e^\lambda + 1 - m\right)\frac{d\lambda}{d\eta} + \frac{1}{(1 - m)^2} \times (1 - e^\lambda)^2 - \frac{1}{(1 - m)^2}(1 - e^\lambda) - 2m = 0. \quad (36)$$

In the following analysis we will consider two classes of solutions of Eq. (36).

A. The case of the constant λ

Equation (36) admits an exact solution of the form $\lambda = \text{constant}$. By denoting $1 - e^\lambda = \delta$, it follows that considering a first order approximation in m , then δ satisfies the following second order algebraic equation

$$\delta^2 - \delta - 2m = 0, \quad (37)$$

which has only one physical solution, namely, $\delta = -2m$ (the other solution contradicts the condition $e^\lambda \geq 1$, which essentially represents the positivity of the mass). Therefore, in the first order approximation the metric coefficient e^λ in the “dark matter” region becomes

$$e^\lambda \approx 1 + 2v_{\text{tg}}^2, \quad e^{-\lambda} \approx 1 - 2v_{\text{tg}}^2. \quad (38)$$

Equation (38) has a straightforward physical interpretation. Since in the Newtonian approximation $v_{\text{tg}}^2 = GM(r)/r$, where $M(r)$ is the total mass of the galaxy, we obtain the metric in a form which is very similar to the Schwarzschild solution of general relativity, i.e., $e^{-\lambda} \approx 1 - 2GM(r)/r$. On the other hand, since $v_{\text{tg}}^2 = \text{constant}$, the mass within the radius r must increase so that $M(r) \sim r$. This “mass”, which is linearly increasing with the distance, is usually interpreted as due to the presence of the dark matter.

In the present approach, the effect of the appearance of a dark “mass” is of a purely geometric origin, resulting from the modification of the basic equations of the gravitational field. Nevertheless, one can formally define a mass $M(r)$ for the dark matter, despite the fact that the origin of this “mass” cannot be related to physical particles. Thus, the rotational galactic curves can be naturally explained in $f(R)$ gravity models without introducing any additional hypothesis. The galaxy is embedded in a modified spherically symmetric geometry, generated by the non-zero contributions of the modified gravitational action. The extra-terms act as a “matter” distribution outside the galaxy.

Once the metric coefficients are known, Eq. (33) provides the function $u(\eta)$ to first order in m as

$$u(\eta) = \frac{1+m}{2} + \frac{\sqrt{1+10m}}{2} \tanh \left[\frac{\sqrt{1+10m}(\eta - \eta_1)}{2} \right], \quad (39)$$

where η_1 is an arbitrary integration constant. The function $F(\eta)$ can be found as

$$F(\eta) = F_0 \exp \left(\frac{1+m}{2} \eta \right) \cosh \left[\frac{\sqrt{1+10m}(\eta - \eta_1)}{2} \right]. \quad (40)$$

For large r , by taking one arbitrary integration constant equal to zero, without a significant loss of generality, so that we keep only the decreasing term in the expression of F , we obtain

$$F(r) = C_1 r^{-2m}, \quad (41)$$

where C_1 is an arbitrary constant. Finally, from Eqs. (20) and (21) we find

$$f(R) = f_0 R^{1+m}, \quad (42)$$

where f_0 is a constant.

B. First order corrections and general properties of λ

In order to obtain a better description of the astrophysical observations at the galactic scale, and also taking into account the discussion of the previous Section, we may also consider a first order correction of the metric coefficient $\exp(\lambda)$, by assuming that λ is small. Hence we may approximate the term $\exp(\lambda)$ as $\exp(\lambda) \simeq 1 + \lambda$, where $\lambda \ll 1$. Moreover, the condition $5/2 \gg 3\lambda/2 - m$ is also satisfied. Therefore, by neglecting the quadratic term λ^2 , Eq. (36) takes the form of an ordinary linear second order inhomogeneous differential equation, which can be written as

$$(1-m)^2 \frac{d^2 \lambda}{d\eta^2} - 5(1-m) \frac{d\lambda}{d\eta} + 2\lambda - 4m = 0, \quad (43)$$

and with the general solution given by

$$\lambda(\eta) = 2m + C_+ \exp(s_+ \eta) + C_- \exp(s_- \eta), \quad (44)$$

where C_{\pm} are arbitrary constants of integration, and

$$s_{\pm} = \frac{5 \pm \sqrt{17}}{2(1-m)}. \quad (45)$$

Hence, to first order in both m and λ , the metric tensor coefficient $\exp(\lambda)$ can be represented as

$$e^{\lambda} = 1 + 2m + C_+ r^{s_+} + C_- r^{s_-}. \quad (46)$$

However, in this case the Riccati equation for u , Eq. (28), cannot be solved exactly, and numerical methods are needed to further investigate the physical properties of this solution.

By introducing a new variable $v = d\lambda/d\eta$, then Eq. (36) can be transformed to a first order differential equation of the form

$$\frac{1}{2} v \frac{dv}{d\lambda} - \frac{1}{1-m} \left(\frac{3}{2} e^{\lambda} + 1 - m \right) v + \frac{1}{(1-m)^2} \times (1 - e^{\lambda})^2 - \frac{1}{(1-m)^2} (1 - e^{\lambda}) - 2m = 0. \quad (47)$$

With the help of the transformation $w = 1/v$ and by introducing a new independent variable $\theta = \exp(\lambda)$, we obtain

$$\frac{dw}{d\theta} + \frac{2}{1-m} \left(\frac{3}{2} + \frac{1-m}{\theta} \right) w^2 - \frac{2}{\theta} \left[\frac{1}{(1-m)^2} \times (1 - \theta)^2 - \frac{1}{(1-m)^2} (1 - \theta) - 2m \right] w^3 = 0. \quad (48)$$

Hence, Eq. (36) has been transformed to a first order nonlinear second kind Abel differential equation of the form $dw/d\theta = A(\theta)w^3 + B(\theta)w^2$ [37]. As it is known from the theory of the Abel differential equations, an equation of this form has an exact solution if and only if the condition $d[A(\theta)/B(\theta)]/d\theta = kB(\theta)$ is satisfied, where k is a constant [37]. A simple calculation using the explicit forms of the functions $A(\theta)$ and $B(\theta)$ corresponding to Eq. (48) shows that this condition cannot be satisfied for all values of m .

Therefore, it follows that the only exact solution of Eq. (36) describing the behavior of the metric coefficient $\exp(\lambda)$ in the dark matter region in $f(R)$ gravity models to first order in the tangential velocity is $\lambda = \text{constant}$. To fully investigate the general behavior of this equation one must use numerical methods.

V. DISCUSSIONS AND FINAL REMARKS

From astrophysical observations and from their interpretation in the framework of the phenomenological

Modified Newtonian Dynamics (MOND) approach [5], it is known that the acceleration needed to explain the observed rotation curves is of the order of 10^{-10} m/s², which can be regarded as only a small deviation from general relativity. Likewise, the Pioneer anomaly, which is also of the order of 10^{-10} m/s², would imply only small modifications of gravity.

On the other hand, recent laboratory tests have confirmed that Newton's second law is in "good agreement" with accelerations of the order of 5×10^{-14} m/s² [38]. Similar constraints have also been obtained for the inverse square law, where it was shown that Newton's law holds down to a length scale of $56 \mu\text{m}$ [39]. Hence, it follows from these observations and experiments that in the R^n modified theories of gravity the parameter n should be of the form $1 + \epsilon$ with $\epsilon \ll 1$.

In the present paper, we have considered the gravitational field equations in $f(R)$ modified theories of gravitation in the flat rotation curves region. Although this is a rather strong assumption, it is valid for at least a significant region of the total velocity profile. As expected, the deviations from standard general relativity are only mild and the exponent of the power law modified gravity takes the form $n = 1 + v_{\text{tg}}^2$. With the use of Eq. (42) we obtain that this theory is defined by the action

$$S = \int f_0 R^{1+v_{\text{tg}}^2} \sqrt{-g} d^4x, \quad (49)$$

where f_0 is a positive constant, given in terms of the tangential velocity, which can be obtained by using Eqs. (20) and (21), and v_{tg}^2 is a number of the order of 10^{-6} . The function $f(R)$ for this modified gravity model can also be approximated as $f(R) = f_0 R^{1+v_{\text{tg}}^2} \approx f_0 R (1 + v_{\text{tg}}^2 \ln R)$. Hence the correction terms to the standard Einstein-Hilbert action is logarithmic.

The weak field limit of the $f(R)$ generalized gravity models has been discussed recently, for star-like objects, in [17] and [18], respectively. By assuming that $f(R)$ is an analytical function at the constant curvature R_0 , that $m_\phi r \ll 1$, where m_ϕ is the effective mass of the scalar degree of freedom of the theory, and that the fluid is pressureless, the post-Newtonian potentials $\Psi(r)$ and $\Phi(r)$ are obtained for a metric of the form $ds^2 = -[1 - 2\Psi(r)] dt^2 + [1 + 2\Phi(r)] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$. One may then find the behavior of $\Psi(r)$ and $\Phi(r)$ outside the star. The analysis leads to a value of $\gamma = 1/2$ for the Post-Newtonian parameter γ , which from Solar System observations is known to have a value of $\gamma = 1$. This result rules out most of the $f(R)$ type modified gravity models. However, an analysis of a Lagrangian of the form given by Eq. (49), denoted in [17] as $f(R) = (R/\alpha)^{1+\delta}$ has also been considered, and the conclusion is that "...this analysis is incapable of determining whether $f(R) = R^{1+\delta}$ gravity with $\delta \neq 1$ conflicts with Solar System tests" [17]. Therefore, the dark matter model obtained from the action given by Eq. (49) is also consistent with the Solar System tests of general relativity. On the other hand, the results of [16] also indicate

that our model is in agreement with all observations on scales smaller than $10 - 20$ kpc.

Another problem facing the $f(R)$ gravity models is the problem of the stability [17, 18, 25]. On a time scale of $\tau \approx 10^{-26}$ s a "fatal instability" develops when $f''(R) < 0$ [18]. For the Lagrangian given by Eq. (49) we have $f''(R) = f_0(v_{\text{tg}}^2/2)(v_{\text{tg}}^2/2 + 1)R^{v_{\text{tg}}^2/2-1} > 0$. Therefore this type of instability does not develop in the present model.

The mass discrepancy in clusters of galaxies is a second major observational evidence leading to the necessity of considering the existence of dark matter at a galactic and extra-galactic scale. The total mass of a cluster of galaxies can be estimated in two ways. First, by taking into account the motions of its member galaxies, the virial theorem provides an estimate, M_V . Second, the total baryonic mass M may be estimated by considering the total sum of each individual member's mass. The mass discrepancy arises as one generally verifies that M_V is considerably greater than M , with typical values of $M_V/M \sim 20 - 30$ [1].

The virial theorem in $f(R)$ modified gravity was derived, by using the collisionless Boltzmann equation, in [31]. The supplementary geometric terms in the modified Einstein equation provide an effective contribution to the gravitational energy, and the total virial mass, proportional to the effective mass associated with the new geometrical term, may account for the virial theorem mass discrepancy in clusters of galaxies. The Lagrangian of the modified gravity model, which can be obtained in terms of quantities directly related to the physical properties of the clusters, and which can be determined from astrophysical observations, is again of the form of the action in Eq. (49). This shows that a modified gravity model, with an action of the form (49), could give a consistent geometric description of the properties of the "dark matter" on both galactic and extra-galactic scales. Once the main physical parameters of matter at the galactic and extra-galactic scale, like the rotational velocities of the test particles around galaxies, or the intra-cluster gas temperature or the gas density profile, are known, the action of the modified gravity model can be completely obtained from observations, and the viability/non-viability of the model can be directly tested by using galactic and cluster of galaxy data, which may also offer an effective alternative to the Solar System tests.

Another possible physical test of the consistency of our model can be provided by the study of the deflection of light. The observational study of the propagation of light at the galactic or galaxy clusters level and, in particular, the investigation of the deflection of photons passing through the regions where the rotation curves of massive test particles are flat, represents one of the most powerful ways by which one could in principle constrain $f(R)$ gravity as an alternative dark matter model for galactic/extra-galactic astrophysical systems. In the flat velocity curves region, the metric coefficients are given by Eqs. (10) and (38), respectively. The general analysis

of the lensing in the constant velocity region was performed in Ref. [3], and the obtained results can also be applied to the $f(R)$ models. Therefore, lensing effects can in principle discriminate between the $f(R)$ gravity and other dark matter or modified gravity models. By using the results of [3] it follows that the $f(R)$ gravity model predicts slightly smaller gravitational lensing effects in the constant velocity region, as compared to the standard dark matter models.

On the other hand, the resulting deflection angle in the present $f(R)$ gravity model is of the same order of magnitude as the lensing angle in the standard dark matter model, the isothermal sphere model, which is well confirmed by observations. This shows that our solution is consistent with the existing observational data, and that high precision observations may discriminate between the different classes of models proposed to explain the motion of test particles around galaxies.

In a series of papers [26], using R^n gravity models, a modified Newtonian potential of the form

$$\Phi(r) = -\frac{Gm}{2r} \left[1 + \left(\frac{r}{r_c} \right)^\beta \right], \quad (50)$$

was considered, where m is the mass of the particle, r_c a constant and the coefficient β depends on the ‘slope’ parameter n in the modified action. In the weak field slow motion approximation β can be expressed as

$$\beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2}. \quad (51)$$

Using the modified Newtonian potential, given by Eq. (50), it was found that the best fit to 15 low luminosity rotation curves in R^n gravity is obtained for $n = 3.5$ [26] (somewhat lower values, in particular, $n = 2.2$, were obtained in [27] and [28], respectively). These results seem to suggest that a strong modification, i.e., a rather large value of n as opposed to $n = 1$, of standard general relativity is required to explain the observed behavior of the galactic rotation curves. As one can see from Eq. (50), the potential obtained in this approach is still asymptotically decreasing, but the corrected rotation

curve, although not flat, is higher than the Newtonian one, thus offering the possibility of fitting the rotation curves without dark matter. However, as one can see from Eq. (14), the correction term to the Newtonian potential in the “dark matter” dominated region, where the rotation curves are strictly flat, must have a logarithmic dependence on the radial coordinate r . This correction term does not appear in Eq. (50), where a power law modified Newtonian potential is assumed to describe the observed behavior of the galactic rotation curves. The difference is also related to the asymptotic behavior of the metric tensor components in the two models. These differences in the Newtonian limit in the two models result in different values of the parameter n in the power-law modified action of the gravity. On the other hand we have to mention that in our approach we have completely neglected the effect of the baryonic matter on the space-time geometry.

In conclusion, we have found that regions with exactly flat galactic rotations curves do not require any kind of dark matter. The rotation curves are a consequence of the additional geometrical structure provided by the modified gravity theories, and a very slight modification of the Einstein-Hilbert Lagrangian may account for the existence of “dark matter”. One can, in principle, rewrite the resulting field equations in terms of the Einstein tensor and interpret the remaining term as a geometrical energy-momentum tensor. The presence of these higher order terms seems to provide us with an elegant geometric interpretation of the dark matter problem.

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